Swami Ramanand Teerth Marathwada University,
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"Some Existence and Uniqueness Results in Fractional Differential and Fractional Difference Equations with its Applications"

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Submitted by

Mr.Rajratana Maroti Kamble
Shri Vitthal Rukhmini College
Sawana, Tq. Mahagoan, Dist.Yavatmal
(M.S.), India -445205
kamblerajratna2@gmail.com

Under the supervision of

Dr. Zampalwad S.S
Head, Department of Mathematics, Gramin
Mahavidyalaya, Vasantnagar, Tq. Mukhed Dist.
Nanded (M.S.), India
sambhaji4267@gmail.com

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Abstract:

We discuss standard approaches to the problem of fractional derivatives and fractional integrals (simply called differintegrals), namely the Riemann-Liouville, the Caputo and the sequential approaches. We prove the basic properties of differintegrals including the rules for their compositions and the conditions for the equivalence of various definitions. Further, we give a brief survey of the basic methods for the solving of linear fractional differential equations and mention limits of their usability. In particular, we formulate the theorem describing the structure of the initial-value problem for linear two-term equations. Finally, we consider some physical applications, in particular fractional advection- dispersion equation and the viscoelasticity problem. Dealing with the first issue we derive the fundamental solution in the form of Levy stable distribution density and then we discuss relations between the generalized central limit theorem and the choice of the corresponding fractional model. In section concerning viscoelasticity we mention some typical models generalizing the standard ones via the replacement of the classical terms by the fractional terms. In particular, we derive the step response functions of those generalized systems.

Similarly we observe that different reasonable definitions for fractional differences are not, in general, equivalent. From the point of view of analysis of boundary value problems, one of the more useful notions until now corresponds to those given by Atici and Eloe. Indeed, there is a good number of papers dealing with discrete fractional boundary value problems using their definition.

A problem that naturally arises, when one intends to study the subject of fractional difference equations, is the correct use of the definition of fractional difference. The reason seems to be in the own definition of fractional sum introduced by Miller and Ross. As remarked by Holm, the definition involves different domains of the functions, which produces confusion and inconsistencies in the applications to fractional difference equations. Moreover, difficulties of

notation, make the statement of some results unwieldy. The main purpose of this talk is to point out that some of the different definitions of fractional difference operators existing in the literature are related by simple translation. This will be done by a correct interpretation of the operators of fractional sums and fractional differences involved

I. Introduction to proposed research topic:

This Synopsis gives a brief description of the literature carried out towards accomplishment of the research work embedded in this thesis in the present form. In the subsequent part of the chapter we have mentioned the review process, the objective outlines and the problem statement of our study.

In recent years, fractional order calculus has been studied as alternative calculus in mathematics and it was applied in various field in science and engineering. Numerous problems in Physics, Chemistry, Biology and Engineering can be models with fractional derivatives. Now the most of modeled fractional differential equations donot have analytic solutions. Therefore, numerical techniques are used intensively, recently the adomian decomposition method, homotopy perturbation method and variational iterative methods have been used for solving wide range of problems. In our work we shall develop new iterative methods such as fractional order Adomian decomposition method, fractional order variational iteration method and explicit fractional order finite difference methods to obtain the numerical solutions of fractional differential and difference equations.

Definition of the Riemann Liouville Fractional Integral:

The Riemann–Liouville definition. For $\alpha \in [n-1,n)$, the α-derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

Definition of the Caputo's Fractional Derivative :

Caputo definition. For $\alpha \in [n-1,n)$, the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_a^{\alpha} \frac{f^{(n)}(x)}{(\mathrm{t}-\mathrm{x})^{\alpha-\mathrm{n}+1}} dx$$

We can introduce the notion of fractional difference in the sense of Caputo for the different notions of fractional sums

Let α > 0. The α-th fractional Caputo like difference is defined by

$$(a)\ C\Delta_a^{\alpha} f\ (t) = \ \Delta_a^{-(m-\alpha)}(\Delta_a^m f\ )(t), \qquad t \in N_{a+(m-\alpha)};$$

$$(b) CGZ\Delta_a^{\alpha} f\ (t) = \ GZ\Delta_a^{-(m-\alpha)}(\Delta_{a+1}^m f)(t), \ \ t\ \in\ N_{a+1};$$

$$(c) C\Delta_a^{\alpha} f\ (t) = \ \Delta_a^{-(m-\alpha)}(\Delta_a^m f)(t), \quad t\ \in\ N_a;$$

Where $m-1 < \alpha < m, \ m\ = \lceil \alpha \rceil$.

We define the fractional difference (in the sense of Caputo) of order α > 0 by $C\Delta^{\alpha} f(n) := \ \Delta^{-(m-\alpha)}(\Delta^m f\ )(n), \quad n\ \in N_0,$ where $m-1 < \alpha < m, m\ =\ \lceil \alpha \rceil$.

II. Purpose of carrying out proposed research work:

Differential equations play a major role in Mathematics, Physics and Engineering. The purpose of the research will develop new iterative methods of fractional differential equations to obtain their solutions. Also, we will represent graphically. The solution of fractional differential equations obtains by new methods using mathematical software Mathematica.

Also we will study Numerical methods to obtain the solutions of fractional partial differential equations, such as Explicit methods. We will compare the solutions obtain by Decomposition methods and Numerical methods.

The synopsis will be divided into different chapters such as introduction of the work, solution of various fractional differential equations, existence and uniqueness theorem, Laplace fractional differential equations applications of fractional differential and difference equations, result and discussion. further,

each chapter will be subdivided into different sections if required .

III. Survey or Review of literature and development in the subject area

Firstly, Miller and Ross and then Gray and Zhang introduced discrete versions of the Riemann-Liouville left fractional integrals and derivatives, called the fractional sums and differences with the delta and nabla operators, respectively. In fractional calculus there are two approaches to obtain fractional derivatives. The first approach is by iterating the integral and then defining a fractional order by using Cauchy formula to obtain Riemann fractional integrals and derivatives.

During the review process we had studied a large number of research papers pertaining to the field of study. However, to summarize, some of the researchers and their contributions in the literature are recorded as under:

S. Browne and J. Bunge (2003) Nussbaum function plays a crucial role in the theory and applications of high gain adaptive control, which has potential advantages in controlling complex and uncertain dynamic systems. Especially in the analysis of universal adaptive stabilization (UAS) problems, without knowing the system parameters, we can still guarantee the asymptotic stability of the system by using the adaptive stabilizers involving Nussbaum functions. MittagLeffler function, which is also called generalized exponential function, is found in the fundamental solutions of fractional-order differential equations. In this paper, based on the properties of Mittag-Leffler functions, authors prove that Mittag-Leffler function is a Nussbaum function under certain conditions and a new family of Nussbaum functions has been found which can be expressed by Mittag-Leffler functions under certain conditions.

K. C. Gupta and T. Gupta (2005) In study of the screening properties of a charged impurity located inside and near the surface of a metal subjected to a magnetic field, there arises an interesting class of Eulerian integrals involving the Bessel function $Jv(z)$ or $J0(z)$, which were expressed in closed forms by

Glasser. To provide generalizations of these Bessel function integrals of Glasser, a number of Eulerian integrals involving various general class of polynomials, Meijer's G-function and Fox's H-function of one and more variables with general arguments have been established hitherto in the literature. In this paper authors evaluated a unified Eulerian integral whose integrand involves the product of the general class of polynomials, general sequence of functions and the multivariable H- Function having general arguments. As the main integral formula has the potential for specializations, a large number of new and known integrals can easily be obtained from it as its particular cases.

V. B. L. Chaurasia and A. S. Shekhawat (2005) The conventional formulation may fail pertaining to the domain of quantum cosmology but Feynman path integrals apply. Feynman path integrals reformulation of quantum mechanics are more fundamental than the conventional formulation in terms of operators. Feynman integrals are useful in the study and development of simple and multiple variable hypergeometric series which in turn are useful in statistical mechanics. The $\bar{H}$-function pertains the exact partition function of the Gaussian model in statistical mechanics, functions useful in testing hypothesis and several others as its special cases. In this paper, authors discussed certain integral properties of general class of polynomials and the $\bar{H}$ function.

G. Jumarie (2009), Author proposes a (new) definition of a fractional Laplace's transform, or Laplace's transform of fractional order, which applies to functions which are fractional differentiable but are not differentiable, in such a manner that they cannot be analyzed by using the Djrbashian fractional derivative. Main properties of this fractal transformation involving the Mittag- Leffler function and the modified Riemann Liouville definition of fractional integral is given, and the inversion theorem is also obtained

K. Gupta and V. Agrawal (2010), In the paper, authors first establish a general

theorem that gives the image of a modified H-transform in the fractional integral operator involving the multivariable H-function. Next, they deduce two important corollaries involving Wright generalized Bessel function, Mittag-Leffler function, Appell function F1 and the product of Whittaker functions. Several other new and known results can also be obtained from the main theorem

B. Satyanarayana and Y. P. Kumar (2011) Recently, the integrals involving general class of polynomial with H-function are evaluated by some researchers due to their utility in the field of science and engineering. In the present paper we establish the integrals involving the product of two multivariable polynomials, H-function of one variable and H-function of 'r' variables. These integrals are unified in nature and a large number of integrals involving simpler functions and polynomials can be derived as their particular cases. Further it is possible to established more integrals by involving other operators also

P. Agarwal, S. Jain and M. Chand (2011),In the paper, authors evaluated certain new finite integral formulas involving Jacobi polynomials and I-function. These integral formulas are unified in nature and act as key formula from which we can obtain several results as their special cases. The formulas established here are basic in nature and are likely to find useful applications in the field of science and engineering.

V. B. L. Chaurasia and Y. Singh (2011), Authors established an integral concerning the product of generalized Lauricella function and two H-function of several complex variables using the series representation of H-function. Further, by giving suitable values to the parameters, the main integral reduces to F function.

M. Chand (2012), In the paper, the author establish three new theorems, which involves I-function and general class of polynomials. Next, he obtain certain new integrals and expansion formulas by the application of these theorems. By giving suitable values to the parameters, the main integral reduces to Fox's H-function, G-function and generalized wright hypergeometric function, etc. which provide

certain results (known and new).

S. Ali (2012), Very recently, Garg and Mittal have evaluated an interesting new unified integral involving Fox's H-function and deduced a very interesting integral involving hypergeometric function as special case. The aim of this research note is to evaluate three new unified integrals (including one given earlier by Garg and Mittal) involving hypergeometric function in terms of the gamma functions

T. Abdeljawad and F. M. Atici (2012) In recent years, discrete fractional calculus gains a great deal of interest by several mathematicians. Firstly, Miller and Ross and then Gray and Zhang introduced discrete versions of the Riemann-Liouville left fractional integrals and derivatives, called the fractional sums and differences with the delta and nabla operators, respectively. In fractional calculus there are two approaches to obtain fractional derivatives. The first approach is by iterating the integral and then defining a fractional order by using Cauchy formula to obtain Riemann fractional integrals and derivatives. The second approach is by iterating the derivative and then defining a fractional order by making use of the binomial theorem to obtain Grünwald Letnikov fractional derivatives. In this paper, authors followed the discrete form of the first approach via the nabla difference operator. They introduced an alternative definition for the Nabla right fractional difference operator and show that two recent definitions of discrete Nabla fractional sum operators are related. Obtaining such a relation between two operators allows one to prove basic properties of the one operator by using the known properties of the other. Authors show this idea with proving power rule and commutative property of discrete fractional sum operators. Authors also prove summation by parts formulas for the right and left fractional sum and difference operators where they employ the Riemann Liouville definition of the fractional difference. As an application of findings authors formalize Initial value problems for nonlinear fractional difference equations. Further study may be attempted to obtain the solutions of such difference

equations by the new and known methods

A. Faraj, T. Salim, S. Sadek and J. Smail (2013), During the last century and due to its involvement in the problems of physics, engineering, and applied sciences, many authors defined and studied in their research papers different generalizations of Mittag-Leffler type function. This paper is devoted for the study of further properties of the generalized Mittag-Leffler function $E\alpha,,,,q(z)$ with another type of fractional calculus operators called Weyl fractional integral and differential operators. A new integral operator depending on Weyl fractional integral operator and containing $E\alpha,\beta,p\gamma,\delta,q(z)$ in its kernel is defined and studied, namely, its boundedness. Also, composition of Weyl fractional integral and differential operators with the new operator is established.

D. A. Bensona, M. M. Meerschaertb, and J. Reviellea (2013), Fractional derivatives can be viewed either as handy extensions of classical calculus or, more deeply, as mathematical operators defined by natural phenomena. This follows the view that the diffusion equation is defined as the governing equation of a Brownian motion. In this paper, we emphasize that fractional derivatives come from the governing equations of stable Lévy motion, and that fractional integration is the corresponding inverse operator. Fractional integration, and its multi-dimensional extensions derived in this way, are intimately tied to fractional Brownian (and Lévy) motions and noises. By following these general principles, we discuss the Eulerian and Lagrangian numerical solutions to fractional partial differential equations, and Eulerian methods for stochastic integrals. These numerical approximations illuminate the essential nature of the fractional calculus. Basically this paper represents a survey of fractional calculus methods in hydrology with a few twists

J. H. Choi, (2013), By using the general theory of differential sub ordination,

several authors considered some interesting properties of multivalent functions associated with various Fractional integral operators. In this paper the author has derived certain subordination properties of the generalized fractional integral operator $\Im 0, \lambda,,$ by using the technique of differential subordination. The main results have been established in the form of three theorems.

V. Kumar (2013),In the paper the author has established N-fractional calculus of a general class of functions and a product of a general class of functions and Fox's H-function. Convergence of general class of functions is also discussed. The main results provide useful extension and unification of a number of (known or new) results for various special cases of general class of functions and Fox's H-function.

IV. Researchgap:

Most important thing is that many fractional differential equations do not have an analytical solution. Therefore, we develop the numericalmethods to obtain there solutions. Also, to discuss the stability and convergence of the numerical methods.Also wide scope in to obtain exact solution of fractional differential equations by developed new iterative techniques and it will represent graphically.

V. Importance of theStudy:

Fractional calculus is becoming a useful and key tool in the analysis of scientific problems in Physics, Engineering, Biology and Economics, Fractional formalism to be useful in solving practical problems, fractional calculus, integration and differentiation of an arbitrary or fractional order, provides new tools that expand the descriptive power of calculus beyond the familiar integer-order concept of rate of change and area under thecurve.

a) International Status: Many scientists and researchers many more contributed in the development of fractional differential equations and Fractional calculus like, Leibniz, Euler, Lagrance, Laplace, Fourier, Abel,

Liouvielli, Grunwald, Riemann, Riesz, Miller, Ross, Igor Prodlubny, Richtmeyer, Morton, Smith, Anderson, Roache .

Lynn R. Willams (1979) proved the existence of multiple fixed point theorems of positive, completely continuous operators on functional spaces which are used in chemical reactor theory. Nguyen Huu Denn (1993) proved a fixed common point theorem for the pair of mappings satisfying both the Banach contraction principle and Caristi's condition in a complete metric space. He also investigated the class of compactible mappings and establish some fixed point theorems for these mappings on the Meir-Keeler condition. Blazej Szmanda (1996), obtained criterion for the oscillatory and asymptotic behaviour of solutions of nonlinear difference equation.

b) National Status: Indian's also contributed remarkable in the development of fractional calculus like R.K. Sexana, V.Srivastava, K.N. Rai, S. Das, K. Gupta, S. Saha Roy, V.B.L. Chawrasia, D. Kumar, Pradeep Kumar, D. Bahuguna, Aalam, Suneel Kumar, Sunny Chauhan and Pant. S.P. Acharya worked on some results on fixed point in uniform space. S.L. Sing from Haridwar, R.P. Pant from Nainatal, B.C. Dhage from Maharashtra, D.N Chaliszar from Gujrat, and many more from India had worked on fixed point theorems and its various applications.

Regional Status: Many researchers from Maharashtra have been actively involve in the study of fractional calculus. Like Kiran, M. Kolwankar, A.K. Pani, PSV Nataraj, S. Bhalekar, V. Daftardar- Gejji, K.C. Takale, J. Verma, D.B. Dhaigude, K.L. Bondar, V.C. Borkar and S.T. Patil (2010), obtained common fixed point theorems in Dynamic programming. They also obtained fixed point theorems with nonlinear alternative via homotopic mapping, comparison theorems for nonlinear difference equations, existence and comparison results for difference phi-Laplacian boundary value problems. P.U. Chopade, K.L. Bondar, V.C. Borkar and S.T. Patil (2010-2011) obtained common fixed points for

nonexpansive type mapping with application to best approximation, existence of solutions to boundary value problems of functional difference equations. K.L. Bondar, A.B. Jadhav and S.T. Patil (2011-2013) obtained existence of coupled solutions for mixed monotone operator, coupled coincidence point theorem for nonlinear contraction in partially ordered metric space, existence of solutions for nonlinear difference equations with nonlocal conditions, Behavior of solutions of difference equations, local and global existence of solutions for summation equations using Tychnoff's fixed point theorem and some common fixed point theorems for the pairs of weakly compactible mappings.

c) Scope of Study:

a) In recent years, their has been a great deal of interest in fractional differential and difference equations. During the last decade, fractional calculus has been applied to almost every field of Science, Engineering and Mathematics. Many application of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems and astrophysics. There are few iterative methods for obtaining the solution of fractional differential and difference equation (FDE), therefore, we develop new existence and uniqueness results to obtain their solutions and discuss the stability and more results.

b) Although a number of researchers have developed new operators involving several special functions, study of their properties, introduce alternate definitions and generalizations of these operators but still there are several functions like I-function, Aleph function, Mittag-Leffler function etc., involving those, one can define a new one.

c) Several researchers have given Integral representations involving operators, special functions and their products. As a large number of operators and special

functions are defined by the researchers so various integral presentations and integral formulas can be derived by using these new operators and special functions.

d) The application area of special functions and operators is quite large in the field of science and technology. Although a large number of workers have developed several applications of special functions and operators but day today developments of new operators, special functions and their generalizations provide the opportunity to develop applications in various fields like Rheology, Environmental science, Medical science etc.

e) Even many Transformations and their applications has been developed by researchers for solving many problems of science and technology, but an easy, suitable and number of alternate approaches is desired to solve different problems of different domains, so there is a good scope of development in this area.

f) The fractional calculus finds use in many fields of science and engineering including the fluid flow, rheology, quantitative biology, electro-chemistry, scattering theory, diffusion transport theory, chemical physics and statistical probability theory, potential theory and many branches of mathematical analysis, like integral and differential equations, operational calculus and univalent function theory, dynamical systems theory, tuning of PID controllers, heat diffusion, circuit synthesis etc. Virtually no area of classical and modern analysis is left untouched by fractional calculus.

g) A significantly large number of earlier works on the subject of fractional calculus give an interesting account of the theory and applications of fractional calculus operators in many different areas of mathematical analysis like ordinary and partial differential equations, integral equations, special functions, summation of series, etc.

h) The fractional integral operators involving various special functions, have found significant importance and applications in various sub-field of applicable

mathematical analysis.

i) The integral transforms, like Laplace transform, Fourier transform, Sumudu transform, Mellin transform etc. are highly useful in study of a wide variety of problems in Mathematics, Physics, Astronomy as well as in Engineering and Technology.

j) The Gaussian hypergeometric function is of fundamental importance in the theory of special functions. The importance of this function lies in the fact that almost all of the commonly used functions of applicable mathematics, mathematical physics, engineering and mathematical biology are expressible as its special cases.

k) There are significantly large number of integral formulas involving several special functions lying in the literature which find useful applications in the field of science and engineering. These formulas also provide unifications and extensions of several results in the literature and act as key formula from which several new and known results can be derived as special cases.

VI. Objectives of Proposed ResearchWork:

1) To study ordinary and partial differential and difference equations.
2) To study different methods for differential and difference equations which are available in the literature.
3) To study basic concept of fractional calculus.
4) To study fractional differential and difference equation.
5) To develop results for fractional differential and difference equations.
6) More than four scientific papers are expected to be published in international journals. To study recent applications of fractional differential and difference equations.
7) The results will be presented at least at two international conferences with few dedicated workshops, and networking environment and opportunities to

share the scientific results with the European community.

8) Participation to scientific workshops, meetings, specialized training and short-term research visits are expected during the Ph.D. project.
9) The proposed work is to prove that how mathematics is useful in other branches of science. We hope that our research work will be useful in applied sciences and engineering field.
10) Hence, in the present work, we will try to develop the new results and new iterative techniques for fractional differential and difference equation and to obtain their solution. These will represented graphically by Mathematica software.

VII. Limitation of the Work:

1) The fractional order model are more adequate than integer order model.
2) Fractional derivatives and integrals enable the description of memory properties of various materials and processes.
3) Most of the fractional differential equations do not have analytical solutions.

VIII. Research Methodology:

The following steps will be including in the methodology.

1) To refer the relevant reference books on differential and difference equation and fractional partial differential equation.
2) To study Mathematica software.
3) To study fractional calculus.
4) To study Iterative / Numerical methods for obtaining approximate solutions

of fractional differentialequations.

5) To discuss the stability and convergence of Iterative / Numerical methods.
6) To represent Numerical solutions of fractional differential and difference equations graphically by Mathematica, Matlab and discuss their nature.

IX. Expected Outcome:

1) We will try to develop the new results and new iterative methods for fractional differential equations.
2) We will try to develop numerical methods for fractional partial differential equations and fractional differential equation.
3) To find the approximate solutions of fractional differential and difference equation by old and new technique and it will represented graphically by Mathematica software.

X. Year wise plan of work and targets to be achieved:

First year plan:

1) To refer the reference books on differential equation and fractional differential equation.
2) To try to find existence and uniqueness results of fractional differential and difference equation .
3) To obtain the solutions of initial and boumdary value problems of fractional differential and difference equation
4) To visualize solutions by graphs using a powerful software Mathematica and Matlab.

Second year plan:

1) To study fractional calculus.
2) To study fractional differential and difference equations.
3) To develop iterative methods for obtain the solution of fractional differential

and difference equations.

4) To visualize the solutions by graphs using Mathematica, Matlab and discuss the nature.
5) To discuss the stability and convergence of iterative methods.

Third year plan :

1) To develop Numerical methods for fractional partial differential equations.
2) To discuss stability and convergence of the numerical methods.
3) We will publish our work in reputed journals.
4) Submission of completed work to concerned authority.

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Mr.Kamble R.M
(ResearchScholar)

Dr. Zampalwad S. S
(ResearchGuide)